A BRIEF CONTROL FOR GENERAL SOLUTIONS OF NORMAL EQUATIONS.*

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If we represent a set of normal equations in the usual manner, we have

$$\begin{aligned}
0 &= [an] + [aa]x + [ab]y + [ac]z + \ldots \\
0 &= [bn] + [ab]x + [bb]y + [bc]z + \ldots \\
0 &= [cn] + [ac]x + [bc]y + [cc]z + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,
\end{aligned}$$

where the equations are continued in similar form until their number m is the same as the number of unknown quantities.

Ordinarily the absolute terms $[an]$, $[bn]$, $[cn]$, . . . have numerical values which are carried through the solution to obtain the numerical values of the unknowns. But if the weights of the values of the unknowns resulting from the solution are desired, it is convenient to retain the symbolic absolute terms through the elimination, and thus obtain the value of each unknown as the sum of a series of terms in $[an]$, $[bn]$, $[cn]$, ... This is called the *general* solution of the equations. Again, if there is reason to suspect that the absolute terms of the observation-equations, from which the normal equations have been formed, will need correction at some future time, it will be convenient to have the general solution. Or it may happen that, before the numerical values of the absolute terms of the normal equations can be given, it is desired to perform the solution as far as the coefficients are concerned. In this last case the general solution is necessary, and it becomes especially desirable to have some brief test of the correctness of the results, while such a test would not be unacceptable whenever for any reason the general solution is made.

The accuracy of the elimination in the general solution might indeed be tested in the same manner as with numerical values of the unknowns, by substituting back in the normal equations. But since the expression for each unknown will have as many terms as there are unknowns, it is evident that, in general, the number of products to be written down will be m^3; and counting only once products that are repeated, it will be found that, in general, the number of products of coefficients to be computed will be $m^3 - \frac{1}{2}m(m-1)$. A briefer control will now be shown.

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For brevity, let the normal equations be written as follows:

$$\begin{array}{l} 0 = N_1 + A_1 x + A_2 y + A_3 z + \dots \\ 0 = N_2 + A_2 x + B_1 y + B_2 z + \dots \\ 0 = N_3 + A_3 x + B_2 y + C_1 z + \dots \\ \dots\dots\dots\dots\dots\dots \\ \qquad\quad \overline{S_1} \qquad \overline{S_2} \qquad \overline{S_3} \dots, \end{array}$$

and let S_1, S_2, S_3, . . . represent respectively the sums of the coefficients in the columns under which they stand. The general solution of the normal equations will give values of $x, y, z,$ . . . which viewed together will be in the normal form, and which may be written as follows:

$$\begin{array}{l} x = K_1 . N_1 + K_2 . N_2 + K_3 . N_3 + \dots \\ y = K_2 . N_1 + L_1 . N_2 + L_2 . N_3 + \dots \\ z = K_3 . N_1 + L_2 . N_2 + M_1 . N_3 + \dots \\ \dots\dots\dots\dots\dots\dots \\ \qquad \overline{S'_1} \qquad \overline{S'_2} \qquad \overline{S'_3} \dots, \end{array}$$

and let S'_1, S'_2, S'_3, . . . represent respectively the sums of the coefficients in the columns under which they stand.

If these values of x, y, z be substituted in the first of the normal equations, we have

$$\begin{array}{l} 0 = N_1 + A_1 K_1 . N_1 + A_1 K_2 . N_2 + A_1 K_3 . N_3 + \dots \\ \qquad + A_2 K_2 . N_1 + A_2 L_1 . N_2 + A_2 L_2 . N_3 + \dots \\ \qquad + A_3 K_3 . N_1 + A_3 L_2 . N_2 + A_3 M_1 . N_3 + \dots \\ \qquad \dots\dots\dots\dots\dots\dots, \end{array}$$

where, after N_1, the first line of the right hand member of the equation comes from the value of x, the second line from the value of y, and so on.

This equation must be generally true; that is, with a given set of coefficients in the normal equations, it must be true whatever the numerical values of N_1, N_2, N_3, . . . may be. From the above equation, therefore, we deduce the following equations, m in number, between the coefficients:

$$\begin{array}{l} -1 = A_1 K_1 + A_2 K_2 + A_3 K_3 + \dots \\ 0 = A_1 K_2 + A_2 L_1 + A_3 L_2 + \dots \\ 0 = A_1 K_3 + A_2 L_2 + A_3 M_1 + \dots \\ \dots\dots\dots\dots\dots\dots, \end{array}$$

Summing these into one equation we have

$$0 = 1 + A_1(K_1 + K_2 + K_3 + \dots) + A_2(K_2 + L_1 + L_2 + \dots)$$
$$+ A_3(K_3 + L_2 + M_1 + \dots) + \dots,$$

or
$$0 = 1 + A_1 S'_1 + A_2 S'_2 + A_3 S'_3 + \dots.$$

Again, if we substitute the values of $x, y, z, \dots$ in the second normal equation, we shall have evidently the same result as from the first equation, except that $A_1, A_2, A_3, \dots$ will be replaced by $A_2, B_1, B_2, \dots$; and similarly with all the remaining normal equations. Therefore we have a set of m equations, as follows:

$$0 = 1 + A_1 S'_1 + A_2 S'_2 + A_3 S'_3 + \dots$$
$$0 = 1 + A_2 S'_1 + B_1 S'_2 + B_2 S'_3 + \dots$$
$$0 = 1 + A_3 S'_1 + B_2 S'_2 + C_1 S'_3 + \dots$$
$$\dots\dots\dots\dots\dots\dots$$

The coefficients of $S'_1, S'_2, S'_3, \dots$ in these equations are identical with the coefficients of $x, y, z, \dots$ in the normal equations, and are arranged in the same manner. Therefore, summing all the above m equations into one final control-equation, we have

$$0 = m + S_1 S'_1 + S_2 S'_2 + S_3 S'_3 + \dots.$$

The *control* consists, therefore, simply in summing up the columns of coefficients as they stand in the normal equations, and also in the general expressions for the unknown quantities; that is, in their expressions as functions of the absolute terms of the normal equations; multiplying these sums together in corresponding pairs; and adding algebraically the sum of the products to the number of the unknowns to produce the result 0.

If the s-control has been employed in forming the normal equations we have

$$[as] = [aa] + [ab] + [ac] + \dots$$
$$[bs] = [ab] + [bb] + [bc] + \dots$$
$$\dots\dots\dots\dots\dots\dots,$$

which will be true of the numerical values in any particular example according to the exactness with which the computation has been made. The right hand members of the above equations will be the exact values of $S_1, S_2, \dots,$ while the $[as]$, $[bs]$, ... will, in general, be close approximate values of these sums.

For a numerical example we may take the following:

NORMAL EQUATIONS.

$$
\begin{array}{llll}
0 = -\ 88 & +\ 27x & +\ 6y & \\
0 = -\ 70 & +\ 6x & +\ 15y & +\ z \\
0 = -\ 107 & & +\ y & +\ 54z \\
\hline
 & S_1 = +\ 33 & S_2 = +\ 22 & S_3 = +\ 55
\end{array}
$$

We have here nothing to do with the numerical terms -88, -70, and -107, but only with their symbols N_1, N_2, N_3.

GENERAL SOLUTION.

$$
\begin{array}{lll}
x = -\ 0.0407N_1 & +\ 0.0163N_2 & -\ 0.0003N_3 \\
y = +\ \ .0163 & -\ \ .0733 & +\ \ .0014 \\
z = -\ \ .0003 & +\ \ .0014 & -\ \ .0186 \\
\hline
S'_1 = -\ 0.0247 & S'_2 = -\ 0.0556 & S'_3 = -\ 0.0175
\end{array}
$$

CONTROL.

$$
\begin{array}{ll}
m & +\ 3. \\
S_1S'_1 & -\ 0.815 \\
S_2S'_2 & -\ 1.223 \\
S_3S'_3 & -\ 0.962 \\
\hline
 & +\ 3.000 \\
 & -\ 3.000 \\
\hline
\text{Sum} & 0.000
\end{array}
$$